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Note

On an Approximation Theorem of Walsh in the p-Adic Field

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The works of Bachman [2], Ahlswede and Bojanic [1], Bojanic [3], Dieudonné [5], Koblitz [6] and Mahler [7] bring out many of the similarities as well as the dissimilarities between analysis over the p-adic field Q_p and analysis over the real field R. If $Z_p = \{x \in Q_p/|x| \leqslant 1\}$ and $f: Z_p \to Q_p$ is a continuous function, then it can be uniformly approximated by polynomials. The above result was originally proved by Dieudonné [5]. Subsequently, Mahler [7] gave constructive proof based on Newton's interpolation formula. A very short proof of the above result was later given by Bojanic [3]. Ahlswede and Bojanic [1] also addressed themselves to such issues as best polnomial approximation. In this short note we prove the p-adic analogue of Walsh approximation theorem which is as follows:

THEOREM 1. *Let $f: Z_p \to Q_p$ be continuous. Let $x_1, x_2, \ldots, x_m$ be a set of m distinct p-adic integers. Then f is uniformly approximable by polynomials h^* that satisfy*

$$h^*(x_k) = f(x_k), \qquad k = 1, 2, \ldots, m. \tag{1}$$

Proof. Let $f_N(x)$ be any sequence of polynomials that approximates f uniformly on Z_p. In view of Mahler's theorem, we can take, for instance,

$$f_N(x) = \sum_{k=0}^{N} a_k(f) \binom{x}{k}, \tag{2}$$

where

$$a_n(f) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} f(k). \tag{3}$$

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For any $t \in J$, choose $N_t \in J$ such that

$$|f(x) - f_N(x)|_p \leqslant p^{-t} \tag{4}$$

for any $n \geqslant N_t$ and $x \in Z_p$. Let $x_1, x_2, \ldots, x_m$ be m distinct points in Z_p and

$$l_{k,m}(x) = \prod_{\substack{i=1 \\ i \neq k}}^{m} \frac{x - x_i}{x_k - x_i}, \qquad k = 1, 2, \ldots, m, \tag{5}$$

be the fundamental polynomials of Lagrange interpolation. The polynomial

$$h_N(x) = \sum_{k=1}^{m} (f(x_k) - f_N(x_k))\, l_{k,m}(x) + f_N(x), \tag{6}$$

where $m \leqslant N + 1$ clearly satisfies the conditions

$$h_N(x_j) = f(x_j), \qquad j = 1, 2, \ldots, m, \tag{7}$$

and

$$|h_N(x) - f(x)|_p \leqslant p^{-t}\{1 + \max_{\substack{t \in Z_p \\ 1 < k < m}} |l_{k,m}(x)|_p\}. \tag{8}$$

To estimate $|l_{k,m}(x)|_p$ it is sufficient to observe that $|x - x_i|_p \leqslant 1$ for all $x \in Z_p$ and that

$$\min\{|x_i - x_j|_p: 1 \leqslant i, j \leqslant m, i \neq j\} = p^{-M} \tag{9}$$

for some $M \in J$. Hence

$$|l_{k,m}(x)|_p \leqslant \prod_{\substack{i=1 \\ i \neq k}}^{m} \leqslant p^{mM}. \tag{10}$$

Using this estimate we obtain from (8) that

$$|h_N(x) - f(x)| \leqslant p^{-t+mM}, \qquad \text{for all } x \in Z_p \text{ and } n \geqslant N_t, \tag{11}$$

and thus the theorem is proved.

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References

1. R. Ahlswede and R. Bojanic, Approximation of continuous functions in p-adic analysis. *J. Approx. Theory* **15** (1975), 190–205.
2. G. Bachman, "Introduction to p-Adic Analysis and Valuation Theory," Academic Press, New York, 1964.
3. R. Bojanic, A simple proof of Mahler's theorem on the approximation of functions of a p-adic variable by polynomials, *J. Number Theory* **6** (1974), 412–415.
4. F. Bruhat, Lectures on some aspects of p-adic analysis, TIFR Lecture Notes, Bombay, 1963.
5. J. Dieudonné, Sur les fonctions continues p-adiques, *Bull. Sci. Math.* **68**(2) (1944), 79–95.
6. N. Koblitz, "p-Adic Numbers, p-Adic Analysis and Zeta-Functions," Graduate Texts in Mathematics, No. 58, Springer-Verlag, Heidelberg, 1977.
7. K. Mahler, "p-Adic Numbers and Their Functions," Cambridge tracts in Mathematics, No. 76, Cambridge Univ. Press, London/New York, 1981.